Associative Brackets - Solution

Mu Games

For this proof we will be working in the framework of monoidal categories[1]. Specifically we will be using the free monoidal category with one generator $\mathbb{B}$, the generator is denoted by X. For this problem we will only look at the subcategory consisting of the objects that don't have the identity object in them and morphisms that don't contain the identity laws.

Objects in this category correspond to bracketings as given in the problem and morphisms correspond to sequences of applications of the associativity axiom.

Now by the proof of Mac Lane's coherence theorem we know that all morphisms with the same source and target can be rewritten into each other. The proof of this uses the following commutative diagrams for all $f\colon A \to X, g\colon B \to Y, h\colon C \to Z$, so these are sufficient to rewrite any path into any other.

$$\begin{tikzcd}
W \otimes (X \otimes (Y \otimes Z)) \arrow[r, "\alpha_{W,X,Y\otimes Z}"] \arrow[d, "1\otimes\alpha_{X,Y,Z}"'] & (W \otimes X) \otimes (Y \otimes Z) \arrow[r, "\alpha_{W\otimes X,Y,Z}"] & ((W \otimes X) \otimes Y) \otimes Z \\
W \otimes ((X \otimes Y) \otimes Z) \arrow[rr, "\alpha_{W,X\otimes Y,Z}"'] & & (W \otimes (X \otimes Y)) \otimes Z \arrow[u, "\alpha_{W,X,Y}\otimes 1"']
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes (B \otimes C) \arrow[r, "f\otimes(g\otimes h)"] \arrow[d, "\alpha"'] & X \otimes (Y \otimes Z) \arrow[d, "\alpha"] \\
(A \otimes B) \otimes C \arrow[r, "(f\otimes g)\otimes h"'] & (X \otimes Y) \otimes Z
\end{tikzcd}$$

The first diagram corresponds to the equation $(\alpha \otimes 1) \circ \alpha \circ (1 \otimes \alpha) = \alpha \circ \alpha$, and the second to $\alpha \circ (f \otimes (g \otimes h)) = ((f \otimes g) \otimes h) \circ \alpha$. When viewed as rewrite rules we note that only the first of these can decrease the number of times associativity is used. From this it follows that we get a string of minimal length by applying associativity on the outer layers first before moving to the sub expressions.

Now it follows that a minimal path is like the one described above and so we can just count the length of that path to get our answer.

[1] Monoidal categories on Wikipedia